ON A CERTAIN CLASS OF EXACT PARTICULAR SOLUTIONS OF THE SHORT-WAVE EQUATIONS

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A class of exact particular solutions of the short-wave equations is investigated; the class in question generalizes the familiar solutions in [1, 2]. The solutions are classified. Examples of new solutions are cited.

Flows with small but abrupt changes in the flow parameters occurring in a narrow zone near the front of a propagating shock wave are described by the system of short-wave equations derived in [3]. The system of short waves, which constitutes a nonlinear system of equations of mixed type, is in a certain sense similar to the system of equations for steady transonic gas motions; unlike the latter system, however, it cannot be transformed into a system of linear equations. This fact complicates the mathematical statement and solution of the short-wave equations considerably, making it necessary to construct exact particular solutions with certain properties associated with a given class of physical problems. Such solutions have been obtained in various artificial ways in each specific case [2 – 5]. The most general of the known solutions is the class of exact solutions derived in [1]. An example of a solution not belonging to the class obtained in [1] is constructed in [2].

1. The short-wave equations in the case of two-dimensional quasisteady flows of a perfect gas are of the form [3]

$$2(\mu-\delta)\frac{\partial\mu}{\partial\delta}+\frac{\partial\nu}{\partial Y}+2k\mu=0,\qquad \frac{\partial\nu}{\partial\delta}-\frac{\partial\mu}{\partial Y}=0 \tag{1.1}$$

Here $k={}^1/_2$ in the case of plane-parallel flows and $k=1$ for axisymmetric flows; the dimensionless functions μ, ν, δ, Y are related to the projections of the velocity on the radius vector u and on its perpendicular v, and to the components of the polar system r, ϑ by the equations

$$u=a_0M_0\mu=a_0M,\qquad v=a_0M_0\sqrt{{}^1/_2(\varkappa+1)\,M_0}\,\nu$$
$$r=a_0t\,[1+{}^1/_2(\varkappa+1)\,M_0\delta],\qquad \vartheta=\sqrt{{}^1/_2(\varkappa+1)\,M_0}\,Y \tag{1.2}$$

System (1.1) clearly admits of the following continuous group of finite transformations:

$$\delta^\circ=a^2(\delta-c),\quad Y^\circ=a(Y-b) \tag{1.3}$$
$$\mu^\circ(\delta^\circ,\ Y^\circ)=a^2(\mu(\delta,\ Y)-c),\quad \nu^\circ(\delta^\circ,\ Y^\circ)=a^3(\nu(\delta,\ Y)+2kcY-d)$$

where a, b, c, d are arbitrary constants which do not alter the form of the equations.

Thus, if some solution $\mu^\circ=\mu^\circ(\delta^\circ, Y^\circ)$, $\nu^\circ=\nu^\circ(\delta^\circ, Y^\circ)$ of system (1.1) is known, then all the equivalent solutions are obtainable from it by way of the formulas

$$\mu(\delta,\ Y)=a^{-2}\mu^\circ(a^2(\delta-c),\ a(Y-b))+c$$
$$\nu(\delta,\ Y)=a^{-3}\nu^\circ(a^2(\delta-c),\ a(Y-b))-2kcY+d \tag{1.4}$$

Formulas (1.4) enable us to construct the class of equivalent solutions with wider properties (see [6] for information on the group properties of (1.1)). Particularly useful are solutions $\mu^\circ=\mu^\circ(\delta^\circ, Y^\circ)$, $\nu^\circ=\nu^\circ(\delta^\circ, Y^\circ)$ of polynomial form in δ or Y, where the solutions constructed with the aid of (1.4) satisfy one of the conditions

$$\mu(\delta, Y) = D \quad \text{for} \quad \delta = c \tag{1.5}$$

$$\nu(\delta, Y) = B \quad \text{for} \quad Y = b \tag{1.6}$$

In many problems of short-wave theory [1 – 5] conditions (1.5), (1.6) with the appropriate constants D, B, c, b play the role of the necessary boundaray conditions which the solutions of (1.1) must satisfy.

2. Let the class of solutions of system (1.1) be defined by the biparametric representation

$$\mu = \sum_{n=0}^{\alpha} \varphi_n(\xi)\eta^n, \qquad \nu = \sum_{n=0}^{\beta} \psi_n(\xi)\eta^n$$
$$\delta = \sum_{n=0}^{\gamma} \chi_n(\xi)\eta^n, \qquad Y = \sum_{n=0}^{\omega} \nu_n(\xi)\eta^n \tag{2.1}$$

Here $\alpha, \beta, \gamma, \omega$ are natural numbers, and the functions $\varphi_\alpha, \psi_\beta, \chi_\gamma, \nu_\omega$ differ from zero. System (1.1) in the variables ξ, η is of the form

$$2(\mu-\delta)\left(\frac{\partial\mu}{\partial\xi}\frac{\partial Y}{\partial\eta}-\frac{\partial\mu}{\partial\eta}\frac{\partial Y}{\partial\xi}\right)+\left(\frac{\partial\nu}{\partial\eta}\frac{\partial\delta}{\partial\xi}-\frac{\partial\nu}{\partial\xi}\frac{\partial\delta}{\partial\eta}\right)+$$
$$+2k\mu\left(\frac{\partial\delta}{\partial\xi}\frac{\partial Y}{\partial\eta}-\frac{\partial\delta}{\partial\eta}\frac{\partial Y}{\partial\xi}\right)=0 \tag{2.2}$$
$$\frac{\partial\mu}{\partial\eta}\frac{\partial\delta}{\partial\xi}-\frac{\partial\mu}{\partial\xi}\frac{\partial\delta}{\partial\eta}=\frac{\partial\nu}{\partial\xi}\frac{\partial Y}{\partial\eta}-\frac{\partial\nu}{\partial\eta}\frac{\partial Y}{\partial\xi}$$

The planes δY and $\xi\eta$ are in one-to-one correspondence if the Jacobian

$$D(\delta, Y)\,/\,D(\xi, \eta) \neq 0$$

and is bounded.

Substituting (2.1) into (2.2) and equating the sum coefficients of equal powers of η in each equation, we obtain a system of ordinary differential equations of the first kind for determining the unknown functions φ_n, ψ_n, χ_n, ν_n.

The general form of the differential equations of the system obtained in this way (for m powers of η) for the first and second equations of (2.2) is

$$\sum_{l=0}^{m}\left[2(\varphi_{m-l}-\chi_{m-l})g_l+\chi_{m-l}h_l-(m-l+1)\chi_{m-l+1}f_l\right]=0 \tag{2.3}$$

$$\sum_{l=0}^{m}\left[(l+1)(\nu_{l+1}\psi'_{m-l}+\chi_{l+1}\varphi'_{m-l})-(m-l+1)(\nu_l'\psi_{m-l+1}+\chi_l'\varphi_{m-l+1})\right]=0$$

where the primes denote derivatives with respect to ξ.

Here

$$g_l=\sum_{i=0}^{l}\left[(l-i+1)\nu_{l-i+1}\varphi_i'-(i+1)\nu_{l-i}'\varphi_{i+1}\right]$$

$$h_l=(l+1)\psi_{l+1}+2k\sum_{i=0}^{l}(l-i+1)\nu_{l-i+1}\varphi_i$$

$$f_l=\psi_l'+2k\sum_{i=0}^{l}\nu_{l-i}'\varphi_i$$

For fixed values of α, β, γ, ω in (2.1) series (2.3) break off, and the number of equations in the system is finite.

3. To determine the highest indices α, β, γ, ω, and thus to determine the forms of solutions (2.1) admitted by system (1.1) we must consider the problem of the compatibility of the number of unknow functions and the number M of equations comprising system (2.3).

Representation (2.1) formally contains $\alpha+\beta+\gamma+\omega+4$ functions. However, this representation and system (2.2) admit of the transformation group

$$\eta = f(\xi^\circ)\eta^\circ + g(\xi^\circ), \qquad \xi = F(\xi^\circ) \tag{3.1}$$

with arbitrary functions $f(\xi^\circ)$, $g(\xi^\circ)$, $F(\xi^\circ)$ which do not alter the form of (2.1) and (2.2). By suitable choice of functions $f(\xi^\circ)$, $g(\xi^\circ)$, $F(\xi^\circ)$ it is possible to ensure that certain functions are determined in final form for (2.1). We note that in the general case of representation (2.1) the number j of such functions is equal to three. When form (2.1) degenerates (e.g. in the absence of the zeroth power of η) we have $j=2$. Instead of (3.1) in this case (2.1) admits of the transformation

$$\eta = f(\xi^\circ)\eta^\circ, \qquad \xi = F(\xi^\circ)$$

Without limiting generality, we can explain the foregoing by considering representation (2.1) for δ and Y. For example, let

$$\xi^\circ = \chi_Y(\xi) \tag{3.2}$$

and let $f(\xi^\circ)$ and $g(\xi^\circ)$ be chosen in such a way that on substituting (3.1) into (2.1) we obtain the vlaues

$$\overset{\circ}{v}_\omega(\xi^\circ) = 1, \qquad \overset{\circ}{v}_{\omega-1}(\xi^\circ) = 0 \tag{3.3}$$

If we omit the subscripts in expressions (3.1), (3.3), then the formal result of transformations (3.1) consists in the fact that j functions have been determined in representation (2.1) and the number of unknown functions is $\alpha+\beta+\gamma+\omega+4-j$.

To determine the number M of equations in the system of ordinary differential equations obtained from (2.3), we determine the highest powers of η for the various terms in Eqs. (2.2) upon substitution of (2.1) into these equations. This yields three distinct combinations (J_1, J_2, J_3) for the first equation and two combinations (J_4, J_5) for the second,

$$J_1 = 2\alpha+\omega-1, \qquad J_2 = \alpha+\gamma+\omega-1, \qquad J_3 = \beta+\gamma-1 \tag{3.4}$$

$$J_4 = \beta+\omega-1, \qquad J_5 = \alpha+\gamma-1$$

The number M of equations is determined by the sum of maximum powers of η in the first and second equations of (2.2),

$$M = \max\{J_1 J_2, J_3\} + \max\{J_4, J_5\} + 2 \tag{3.5}$$

which brings us to the consideration of six distinct combinations of relationships among exponents (3.4).

The correspondence between the number of unknowns and the number M of system equations is determined by the condition

$$\alpha+\beta+\gamma+\omega+4-j = M+q \tag{3.6}$$

where q is the determinacy coefficient of the system (the system is determined for $q=0$, overdetermined for $q<0$, and underdetermined for $q>0$).

It is interesting to note that the number $i=q+j$ is uniquely determined by the combination α, β, γ, ω in accordance with (3.6) and (3.5).

The case where exponents (3.4) are equal for each equation of (2.2) immediately gives us the solution with

$$\alpha = 2(4-i), \quad \beta = 3(4-i), \quad \gamma = 2(4-i), \quad \omega = 4-i$$

This is analogous to the case considered in [7] for the system of equations of transonic gas flows.

Considering the various combinations of α, β, γ, ω and eliminating the case $\gamma = \omega = 0$, where $D(\delta, Y)/D(\xi, \eta) = 0$, we can generally establish quite readily in accordance with (3.5), (3.6) that the number $i \leqslant 3$. This means that for the general form of solution (2.1) where $j = 3$, the number $q \leqslant 0$, and according to (3.6) we can only have determined and overdetermined systems of ordinary differential equations.

In the case of determined systems of ordinary differential equations for $q = 0$, $i = 3$ the parameters α, β, γ, ω assume the values

$$0 \leqslant \alpha \leqslant 2, \quad 0 \leqslant \beta \leqslant 3, \quad 0 \leqslant \gamma \leqslant 2, \quad 0 \leqslant \omega \leqslant 1 \qquad (3.7)$$

Hence, it is most convenient to classify solutions (2.1) according to γ and ω. This gives rise to four basic combinations of values (γ, ω),

$$(2,1), \ (1,1), \ (0,1), \ (1,0) \qquad (3.8)$$

This corresponds to four types of solutions with distinct values of (α, β)

$$(2,3), (1,2), (0, 1) \text{ for } \gamma = 2, \ \omega = 1 \qquad (3.9)$$

$$(1, 2), (1,1), (0,1), (0,0) \text{ for } \gamma = 1, \ \omega = 1 \qquad (3.10)$$

$$(0,1), (0,0) \text{ for } \gamma = 0, \ \omega = 1 \qquad (3.11)$$

$$(2,3), (1,2), (1,1), (0,1), (0,0) \text{ for } \gamma = 1, \ \omega = 0 \qquad (3.12)$$

4. Let us consider the problem of using transformation (3.1) for the resulting types of solutions (3.8). The form of the equations of system (2.3) is simplest when the functions $f(\xi^\circ)$, $g(\xi^\circ)$, $F(\xi^\circ)$ are chosen on the basis of the form of the representations for Y and δ.

Then for $\omega = 1$ for

$$Y = \nu_0(\xi) + \nu_1(\xi)\eta, \quad \delta = \sum_{n=0}^{\gamma} \chi_n(\xi)\eta^n \qquad (\gamma = 0, 1, 2)$$

setting $\xi^\circ = \chi_\gamma(\xi)$ (the inverse function $\xi = F(\xi^\circ)$),

$$f(\xi^\circ) = 1/\nu_1(\xi^\circ), \quad g(\xi^\circ) = -\nu_0(\xi^\circ)/\nu_1(\xi^\circ),$$

recalling result (3.3), and omitting the indices, we find that

$$Y = \eta, \qquad \delta = \sum_{n=0}^{\gamma-1} \chi_n(\xi) Y^n + \xi Y^\gamma \quad (\gamma = 0,1,2) \qquad (4.1)$$

Transformation (3.1) is nondegenerate ($\nu_1 \neq 0$ for $\omega = 1$) if $\chi_\gamma'(\xi) \neq 0$ and is bounded. Thus, in the case $\omega = 1$, if $\chi_\gamma(\xi)$ is not a constant, we have (4.1) and solutions (2.1) of polynomial form in Y.

In the case $\gamma = 1$ for

$$\delta = \chi_0(\xi) + \chi_1(\xi)\eta, \qquad Y = \sum_{n=0}^{\omega} \nu_n(\xi)\eta^n \qquad (\omega = 0, 1)$$

we obtain, as in the previous case,

$$\delta = \eta, \qquad Y = \sum_{n=0}^{\omega-1} \nu_n(\xi)\delta^n + \xi\delta^\omega \qquad (4.2)$$

Transformation (3.1) is nondegenerate ($\chi_1 \neq 0$ for $\gamma = 1$) if $\nu_\omega'(\xi) \neq 0$ and is bounded. Hence, in the case $\gamma = 1$, if $\nu_\omega(\xi)$ is not a constant, we have (4.2), and solutions (2.1) are of polynomial form in δ.

Thus, determination of solutions of the form (2.1) and the use of transformation (3.1) yields solutions for which one of the functions ($\chi_\gamma(\xi)$, $\nu_\omega(\xi)$ in the cases $\omega = 1, \gamma = 1$) is not a constant. Since the form of the solution is not known in advance, one can generally assume the reverse, i.e. one can verify the existence of solutions with a constant value of this function in each specific case. If such solutions, in fact, do exist, we shall call them "lost" solutions.

5. Let us write out the resulting system of differential equations in general form. This basic system of equations is obtainable from (2.3) in accordance with (3.7) for $\alpha = 2$, $\beta = 3, \gamma = 2, \quad \omega = 1$

$$\begin{gathered}
(\varphi_2 - \chi_2) K_2 - 2\chi_2 G_3 + \chi_2' H_3 = 0 \\
(\varphi_2 - \chi_2) K_1 + (\varphi_1 - \chi_1) K_2 - 2\chi_2 G_2 + \chi_2' H_2 - \chi_1 G_3 + \chi_1' H_3 = 0 \\
(\varphi_2 - \chi_2) K_0 + (\varphi_1 - \chi_1) K_1 + (\varphi_0 - \chi_0) K_2 - 2\chi_2 G_1 + \chi_2' H_1 - \chi_1 G_2 + \chi_1' H_2 + \\
+ \chi_0' H_3 = 0 \\
(\varphi_1 - \chi_1) K_0 + (\varphi_0 - \chi_0) K_1 - 2\chi_2 G_0 - \chi_1 G_1 + \chi_1' H_1 + \chi_0' H_2 = 0 \\
(\varphi_0 - \chi_0) K_0 - \chi_1 G_0 + \chi_0' H_1 = 0 \\
\psi_3' \nu_1 - 3\psi_3 \nu_1' + 2\varphi_2' \chi_2 - 2\varphi_2 \chi_2' = 0 \\
\psi_2' \nu_1 - 2\psi_2 \nu_1' - 3\psi_3 \nu_0' + 2\varphi_1' \chi_2 - \varphi_1 \chi_2' + \varphi_2' \chi_1 - 2\varphi_2 \chi_1' = 0 \\
\psi_1' \nu_1 - \psi_1 \nu_1' - 2\psi_2 \nu_0' + 2\varphi_0' \chi_2 + \varphi_1' \chi_1' - \varphi_1 \chi_1' - 2\varphi_2 \chi_0' = 0 \\
\psi_0' \nu_1 - \psi_1 \nu_0' + \varphi_0' \chi_1 - \varphi_1 \chi_0' = 0
\end{gathered} \tag{5.1}$$

Here

$$K_i = \varphi_i' \nu_1 - (2 - i)\, 2^{2i-1} \varphi_{i+1} \nu_0' - i \varphi_i \nu_1' \qquad (i = 0, 1, 2)$$

$$G_j = {}^1/_2 \psi_j' + (3 - j)^{j(1-j)(2-j)} k \varphi_j \nu_0' + j^{(1-j)(2-j)(3-j)} k \varphi_{j-1} \nu_1' \qquad (j = 0, 1, 2, 3)$$

$$H_l = l / 2\psi_l + k \varphi_{l-1} \nu_1 \qquad (l = 1, 2, 3)$$

Let us note some of the properties of system (5.1). System (5.1) contains nine equations which can be used for determining nine unknowns provided one specifies three functions ($j = 3$) in accordance with (3.1). System (5.1) remains determined if we convert to the "symmetric" form of the solutions, when

$$\begin{gathered}
\varphi_1(\xi) = \psi_2(\xi) = \psi_0(\xi) = \chi_1(\xi) = \nu_0(\xi) = 0 \\
\mu = \varphi_2(\xi)\eta^2 + \varphi_0(\xi), \qquad \nu = \psi_3(\xi)\eta^3 + \psi_1(\xi)\eta \\
\delta = \chi_2(\xi)\eta^2 + \chi_0(\xi) \qquad Y = \nu_1(\xi)\eta
\end{gathered} \tag{5.2}$$

In fact, we readily perceive that the system itself now contains five equations for five unknowns, since in this case one needs to specify two functions ($j = 2$) only.

Moreover, in the particular case of (5.2) where

$$\mu = \varphi_2(\xi)\eta^2, \quad \nu = \psi_3(\xi)\eta^3, \quad \delta = \chi_2(\xi)\eta^2, \quad Y = \nu_1(\xi)\eta \tag{5.3}$$

for $\varphi_0(\xi) = \psi_1(\xi) = \chi_0(\xi) = 0$ system (5.1) reduces to two equations for determining two functions ($j = 2$).

Similarly, for $\varphi_2(\xi) = \psi_3(\xi) = 0$ we obtain a system of three equations with three unknown functions for solutions of the form

$$\mu = \varphi_0(\xi), \quad \nu = \psi_1(\xi)\eta, \quad \delta = \chi_2(\xi)\eta^2 + \chi_0(\xi), \quad Y = \nu_1(\xi)\eta \tag{5.4}$$

Solutions of symmetric type usually satisfy a condition of type (1.6) and are therefore of particular practical interest.

6. Setting $\alpha = 2$, $\beta = 3$, $\gamma = 2$, $\omega = 1$ in general form (2.1) of the solutions and setting

$$\chi_2(\xi) = \xi, \qquad \nu_1(\xi) = 1, \qquad \nu_0(\xi) = 0 \tag{6.1}$$

we find in accordance with (4.1) that (2.1) and (5.1) yield the familiar class of solutions derived in [2],

$$\begin{gathered}\mu = \varphi_2(\xi)\, Y^2 + \varphi_1(\xi)\, Y + \varphi_0(\xi)\\ \nu = \psi_3(\xi)\, Y^3 + \psi_2(\xi)\, Y^2 + \psi_1(\xi)\, Y + \psi_0(\xi)\\ \delta = \xi Y^2 + \chi_1(\xi) Y + \chi_0(\xi)\end{gathered} \tag{6.2}$$

We also obtain the corresponding system of differential equations. This system is investigated in detail in [1]. Most interesting from the physical standpoint are the symmetric solutions of the form (5.2) – (5.4). Such solutions of form (5.2) have been constructed and used for the solution of problems with a rectilinear boundary (problems of shock-wave reflection from rigid walls and free surfaces) [1, 4, 5].

Solutions of the form (5.3) in the case (6.1) bring us to the case of self-similar solutions of (1.1), namely to

$$\mu = Y^2 \varphi_2(\xi), \qquad \nu = Y^3 \psi_3(\xi), \qquad \xi = \delta / Y^2 \tag{6.3}$$

which, for example for $k = {}^1/_2$, are of the form

$$\begin{gathered}\mu = Y^2\,[c_1\xi - c_1(c_1 - {}^1/_2) + 3c_2\sqrt{2\xi - c_1}]\\ \nu = 2Y^3\,[c_2(\xi - 2c_1)\sqrt{2\xi - c_1} - c_1(c_1 - {}^1/_2)\,\xi + c_3]\end{gathered}$$

Examples of symmetric solutions of the type (5.4) for $k = {}^1/_2$ and $k = 1$ are constructed in [3].

7. Let us investigate the class of "lost" solutions of basic system (5.1) for

$$\chi_2(\xi) = a, \qquad \nu_1(\xi) = 1, \qquad \nu_0(\xi) = 0 \qquad (a = \mathrm{const}) \tag{7.1}$$

of the form

$$\begin{gathered}\mu = \varphi_2(\xi)\, Y^2 + \varphi_1(\xi)\, Y + \varphi_0(\xi)\\ \nu = \psi_3(\xi)\, Y^3 + \psi_2(\xi)\, Y^2 + \psi_1(\xi) Y + \psi_0(\xi)\\ \delta = aY^2 + \chi_1(\xi) Y + \chi_0(\xi)\end{gathered} \tag{7.2}$$

From the first and sixth equations of (5.1) with allowance for (7.1) we obtain $\varphi_2 = c_2$, $\psi_3 = c_3$. For $a \neq 0$, setting

$$\varphi_2 = a(1 - 2a), \qquad \psi_3 = {}^2/_3 a(1 - 2a)(2a - k) \tag{7.3}$$

we find that the second and seventh equations of (5.1) (with allowance for (7.1))coincide. For $a = 0$ the functions $\varphi_2 = 0$, $\psi_3 = 0$, and the second equation of (5.1) is satisfied identically.

In both cases the systems corresponding to (5.1) become determined following the transformation $F(\xi) = \xi^\circ$. Further on we shall take as our $F(\xi)$ one of the functions of system (5.1), which together with (7.1) is equivalent to (3.1).

Let us consider symmetric solutions ($\varphi_1 = \psi_0 = \chi_1 = \psi_2 = 0$) of type (5.2), (5.4) which system (5.1) admits for form (7.2).

For $a \neq 0$, $\chi_0(\xi) = \xi$ for solutions of type (5.2) ((3.9), $\alpha = 2$, $\beta = 3$) ,

$$\mu = \varphi_2(\xi)\, Y^2 + \varphi_0(\xi), \qquad \nu = \psi_3(\xi)\, Y^3 + \psi_1(\xi)\, Y, \qquad \delta = aY^2 + \xi \tag{7.4}$$

system (5.1) reduces to a differential equation for $\varphi_0(\xi)$, namely to

$$(\varphi_0 - \xi)\,\varphi_0' + (k - a)\,\varphi_0 + a(1 - 2a)\,\xi + {}^1/_2 c_1 = 0 \tag{7.5}$$

to formulas (7.3) for φ_2, ψ_3 , and to the following expression for ψ_1 :

$$\psi_1 = 2\xi\varphi_2 - 2a\varphi_0 + c_1 \quad (7.6)$$

Solutions (7.4) in this case are similar to the solutions in [8] for the equations of transonic gas motions. The form $\varphi_0(\xi)$ of solution (7.5) depends on the sign of $D = 9a^2 - 2a(k+1) + (1-k)^2$.

We note, for example, the case $D > 0$, which is of particular practical interest. Here the family of solutions of (7.5) has asymptotes and can be expressed as

$$|u - px|^{p-1}| \cdot |u - qx|^{1-q} = c \qquad (c > 0) \quad (7.7)$$

Here

$$u = \varphi_0(\xi) - A, \quad x = \xi - A, \quad A = {}^1/_2\, c_1 (2a^2 - k)^{-1}, \ a \neq \pm\sqrt{k/2}$$

$$p = {}^1/_2 (a + 1 - k + \sqrt{D}), \qquad q = {}^1/_2 (a + 1 - k - \sqrt{D}) \quad (7.8)$$

The singular solutions $u = px$, $u = qx$ of (7.5) define the asymptotes for family (7.7). One such solution for $k = {}^1/_2$ was used in [2] (in solving the problem of regular reflection from a rigid wall).

In the case $a = \pm\sqrt{k/2}$ the solutions of (7.5) are given by the expressions

$${}^1/_2 c_1 \ln[r(\varphi_0 - \xi) + {}^1/_2\, c_1] = r^2(\xi + c) + r(\varphi_0 - \xi) \quad (7.9)$$

$$\varphi_0 = (1 - r)\,\xi + c_2, \ \varphi_0 = \xi - {}^1/_2 c_1 r^{-1} \ \text{ for } \ r = 1 + k \mp \sqrt{k/2}$$

The particular case of symmetric solutions of the form (5.4) ((3.9), $\alpha = 0$, $\beta = 1$) results from (7.4) for $a = {}^1/_2$, when $\varphi_2 = \psi_3 = 0$. Setting $\varphi_0(\xi) = \xi$, in accordance with (3.1) for the solution of the form

$$\mu = \xi, \quad \nu = [-\xi + c_1]\, Y, \quad \delta = aY^2 + \chi_0(\xi) \quad (7.10)$$

we obtain

$$\chi_0(\xi) = ce^{2\xi/c_1} + \xi + {}^1/_2 c_1 \qquad \text{for } k = {}^1/_2$$

$$\chi_0(\xi) = c(\xi + c_1)^2 + 2(\xi + c_1) - c_1 \quad \text{for } k = 1$$

8. The class of "lost" solutions of (7.2) in the particular case $a = 0$ ($\varphi_2 = \psi_3 = 0$) leads to the cases $\gamma = 1$, $\omega = 1$ (3.10), $\gamma = 0$, $\omega = 1$ (3.11).

We obtain solutions for $\gamma = 1$, $\omega = 1$, $\alpha = 1$, $\beta = 2$ (3.10) of the form

$$\mu = \varphi_1(\xi)\, Y + \varphi_0(\xi), \ \ \nu = \psi_2(\xi)\, Y^2 + \psi_1(\xi)\, Y + \psi_0(\xi), \ \ \delta = \xi Y + \chi_0(\xi) \quad (8.1)$$

by solving system (5.1) with allowance for (7.1) for $a = 0$, $\chi_1(\xi) = \xi$. With allowance for (7.1), we find from the seventh equation of (5.1) that

$$\psi_2 = c_1 \quad (8.2)$$

The third equation of (5.1) with allowance for (8.2) yields the following equation for φ_1:

$$(\varphi_1 - \xi)\,\varphi_1' + k\varphi_1 + c_1 = 0 \quad (8.3)$$

with solutions of the form

$$\varphi_1 = -(2c)^{-1}[4cc_1 + 1 \pm \sqrt{4cc_1 + 1 + 2c\xi}], \ \ \varphi_1 = -2c_1 \quad (8.4)$$

for $k = {}^1/_2$, and of the form

$$(\varphi_1 + c_1)\ln(\varphi_1 + c_1) + c(\varphi_1 + c_1) = \xi + c_1, \quad \varphi_1 = -c_1 \quad (8.5)$$

for $k = 1$.

In all cases where $\varphi_1 \neq -c_1 / k$ system (5.1) reduces to the solution of the following Abel equation of the second kind for φ_0:

$$L(\xi)\,\varphi_0\varphi_0' + \theta(\xi)\,\varphi_0' + K(\xi)\,\varphi_0^2 + N(\xi)\,\varphi_0 + M(\xi) = 0 \quad (8.6)$$

with the coefficients given by

$$L(\xi) = 1 + (k-1)\, g/h, \quad N(\xi) = f\,(g/h)' + kch^{(1-k)/k}, \quad \varphi_1' + k\bar{\chi}_0'$$
$$K(\xi) = k\,(g/h)' \qquad M(\xi) = f\,[ch^{(1-k)/k}\varphi_1' + \bar{\chi}_0']$$
$$\theta(\xi) = {}^1/_2\,\xi^2 + f\,(g/h) - ch^{1/k} - \bar{\chi}_0$$

Here

$$g = \xi - \varphi_1, \qquad h = c_1 + k\varphi_1, \qquad f = {}^1/_2\,(\psi_1 - \xi\varphi_1)$$

and $\bar{\chi}_0(\xi)$ is an arbitrary particular solution of the equation

$$2\,(c_1 + k\varphi_1)\chi_0' - 2\varphi_1'\chi_0 = \xi\psi_1' - \psi_1$$

The functions ψ_1, ψ_0, χ_0 are given by the formulas

$$\psi_1' = \varphi_1 - \xi\varphi_1', \qquad \psi_0' = \varphi_1\chi_0' - \xi\varphi_0' \tag{8.7}$$
$$\chi_0 = c\,(c_1 + k\varphi_1)^{1/k} + (\xi - \varphi_1)(c_1 + k\varphi_1)^{-1}\varphi_0 + \bar{\chi}_0$$

For example let us cite the solution for $k = {}^1/_2$, $\varphi_1 = {}^1/_2\,(\xi - 2c_1)$ (φ_1 results from (8.4) for $c = 0$)

$$\mu = \varphi_0(\xi) + {}^1/_2\,(\xi - 2c_1)\,Y$$
$$\nu = -c_1\,[2\varphi_0(\xi) + c_3(\xi + 2c_1)^2 + c_2] + c_5 + c_3\,({}^1/_3\xi^3 + c_1\xi^2) - (c_1\xi - c_2)\,Y + c_1Y^2 \tag{8.8}$$
$$\delta = 2\varphi_0(\xi) + c_3(\xi + 2c_1)^2 + c_2 + \xi Y$$

Here

$$\varphi_0(\xi) = (\xi + 2c_1)\,[2c_1 \ln(\xi + 2c_1) - {}^1/_2\xi + c_4] - c_2$$

For $\varphi_1 = -c_1/k$ system (5.1) becomes considerably simpler and the solutions can be written in finite form.

More particular forms of solutions (8.1) with $\alpha \leqslant 1$ $\beta < 2$ (3.10) are obtainable in the same way as the above solutions for $c_1 = 0$.

In the case $\gamma = 1$, $\omega = 1$ system (5.1) admits of "lost" solutions for $\chi_1(\xi) = a$ (a — const) which for ($k = {}^1/_2, 1$) are linear functions of δ, Y for μ and ν. For $a = 0$, $\varphi_0(\xi) = \xi$ we arrive at the case $\gamma = 0$, $\omega = 1$.

The solutions for $\gamma = 0$, $\omega = 1$, $\alpha = 0$, $\beta = 1$ (3.11) of symmetric form of type (5.4) are

$$\mu = \xi, \qquad \nu = c_1Y, \quad \delta = \chi_0(\xi) \tag{8.9}$$

Here

$$\chi_0(\xi) = c\,(\xi + c_1)^2 + 2\,(\xi + c_1) - c_1 \quad \text{for } k = {}^1/_2$$
$$\chi_0(\xi) = -{}^1/_2c_1 - (\xi + {}^1/_2c_1)\ln c\,(\xi + {}^1/_2c_1) \quad \text{for } k = 1$$

Solutions (8.9) for $c_1 = 0$ (one-dimensional flows) are similar solutions obtained in [9] and used for solving problems on the decay of weak shock waves.

9. For $\gamma = 1$ representation (2.1) is of polynomial form in δ, and as noted above, the solutions in this case can satisfy conditions of type (1.5) on the line of parabolicity of Eqs. (1.1) (the "sonic" line $\delta = \mu = c$).

To construct the solutions corresponding to $\gamma = 1$, $\omega = 1$ (3.10) for $\nu_1(\xi) = \xi$, $\chi_1(\xi) = 1$, $\chi_0(\xi) = 0$ of the form

$$\mu = \varphi_1(\xi)\,\delta + \varphi_0(\xi), \quad \nu = \psi_2(\xi)\,\delta^2 + \psi_1(\xi)\,\delta + \psi_0(\xi), \quad Y = \xi\delta + \nu_0(\xi) \tag{9.1}$$

we can make use of the above results for solutions of the form (8.1) rewritten in powers of δ.

It is easy to show that the transformation

$$\xi = 1/\xi^\sigma, \qquad Y = \xi^\sigma\,[\delta - \chi_0(1/\xi^\sigma)] \tag{9.2}$$

transforms any solution of the form (8.1) into a solution of the form (9.1). For example, solution (8.8) can be rewritten as

$$\begin{gathered}
\mu = c_1\xi\chi_0(\xi^{-1}) - {}^1/_2 c_3(\xi^{-1}+2c_1)^2 - {}^1/_2 c_2 + {}^1/_2(1-2c_1\xi)\,\delta \\
\nu = c_1\xi^2\chi_0^2(\xi^{-1}) - c_2\xi\chi_0(\xi^{-1}) + c_3({}^1/_3\xi^{-3}+c_1\xi^{-2}) + \\
+ c_5 - (2c_1\xi^2\chi_0(\xi^{-1}) + c_1 - c_3\xi)\,\delta + c_1\xi^2\delta^2 \\
Y = -\xi\chi_0(\xi^{-1}) + \xi\delta
\end{gathered} \tag{9.3}$$

(where the indices have been omitted).

Here

$$\chi_0(1/\xi) = (\xi^{-1}+2c_1)\,[4c_1\ln(\xi^{-1}+2c_1) + (c_3-1)(\xi^{-1}+2c_1) + 2c_4] + 4c_1^2 - c_2$$

In another case, for $\gamma = 1$, $\omega = 0$ (3.12), setting $\nu_0 = \xi$, $\chi_1 = -1$, $\chi_0 = c$, we obtain the solutions of the form

$$\begin{gathered}
\mu = \varphi_2(Y)(c-\delta)^2 + \varphi_1(Y)(c-\delta) + \varphi_0(Y) \\
\nu = \psi_3(Y)(c-\delta)^3 + \psi_2(Y)(c-\delta)^2 + \psi_1(Y)(c-\delta) + \psi_0(Y)
\end{gathered} \tag{9.4}$$

in the general case.

The corresponding system of differential equations obtainable by way of (5.1) reduces to the differential equation

$$\varphi_2'' + 12\varphi_2^2 = 0 \tag{9.5}$$

with the solutions

$$\varphi_2 = -{}^1/_2\,\wp(Y+c_2), \qquad \varphi_2 = 0 \tag{9.6}$$

(where $\wp(Y+c_2)$ is a Weierstrass function with the invariants $g_2 = 0$, $g_3 = c_1$) and to the system of linear differential equations

$$\begin{gathered}
\varphi_1'' + 12\varphi_2\varphi_1 = 4(k-2)\,\varphi_2, \qquad \psi_2 = -{}^1/_2\varphi_1' \\
\varphi_0'' + 4\varphi_2\varphi_0 = 4c\varphi_2 - 2\varphi_1(\varphi_1+1-k), \quad \psi_1 = -\varphi_0' \\
\psi_0' = -2k\varphi_0 + 2(\varphi_0 - c)\,\varphi_1, \quad \psi_3 = -{}^1/_3\varphi_2'
\end{gathered} \tag{9.7}$$

For example, let us cite the solution for $k = {}^1/_2$, $\varphi_2 = -{}^1/_2\,\wp(Y+c_2)$ of the form

$$\begin{gathered}
\mu = -{}^1/_2\,\wp(Y+c_2)(c-\delta)^2 - {}^1/_2(c-\delta) + c \\
\nu = {}^1/_6\,\wp'(Y+c_2)(c-\delta)^3 - c(Y+c_3) + c_4
\end{gathered} \tag{9.8}$$

The solutions for $\alpha \leqslant 1$, $\beta \leqslant 2$ (3.12) are obtainable for $\varphi_2 = 0$ and are of the form

$$\begin{gathered}
\mu = \varphi_1(Y)(c-\delta) + \varphi_0(Y) \\
\nu = -{}^1/_2\,\varphi_1'(Y)(c-\delta)^2 - \varphi_0'(Y)(c-\delta) + \psi_0(Y)
\end{gathered} \tag{9.9}$$

Here ψ_0 can be determined from (9.7),

$$\begin{gathered}
\varphi_1 = c_1 Y + c_2 \\
\varphi_0 = -[{}^1/_6 c_1^2 Y^4 + {}^1/_3 c_1(2c_2+1-k)\,Y^3 + c_2(c_2+1-k)\,Y^2 + c_3 Y + c_4]
\end{gathered} \tag{9.10}$$

Solutions (9.8) are similar in form to the solutions obtained by L. V. Ovsiannikov for the equations of transonic gas motions. We note that if the functions ψ_0, φ_0 are constant in solutions of the form (8.1), (9.1), (9.4) (this can often be achieved by assigning suitable values to the arbitrary constants), then transformation (1.4) enables one to obtain solutions (for example (9.8)) which satisfy conditions (1.5), (1.6).

In conclusion let us consider the applicability of the above solutions to specific problems. Transformation group (1.4) enables us to express most of the resulting solution in a form which satisfies conditions (1.5), (1.6). However, conditions (1.5), (1.6) are merely the simplest of the conditions used in short-wave problems, so that the problem of applicability of a given solution to the investigation of a specific problem depends on whether it is possible to solve the problem not only with allowance for (1.5), (1.6) (or independently of the latter) but also to satisfy the specific conditions (characteristic features) of the given problem.

We note that the proposed method of constructing exact particular solutions can be used in similar fashion in dealing with other nonlinear systems of partial differential equations (e.g. the equations of short waves in a viscous heat conducting gas, the equations of transonic flows of a perfect and viscous gas).

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